UM–P–97/19
To appear in the *American Journal of Physics*

Classical antiparticles

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(29 March 1997)

Abstract

We review how *antiparticles* may be introduced in classical relativistic mechanics, and emphasize that many of their paradoxical properties can be more transparently understood in the classical than in the quantum domain.

I. Introduction

Recently,[1] we reviewed briefly the physics and early history of the Foldy–Wouthuysen transformation,[2,3] emphasizing that the transformed representation is the only one in which a *classical limit* of the Dirac equation can be meaningfully extracted, in terms of particles and antiparticles. But few textbooks actually describe how antiparticles *can* be dealt with in classical mechanics. Discussions of antiparticles usually begin with the "negative energy problem": the inevitable introduction, in relativistic mechanics, of what appears to be a "spurious" set of mirror eigenstates of negative energy; their reinterpretation by Dirac as "holes" in a filled Fermi sea of vacuum electrons; and their further reformulation, in quantum field theory, as completely valid eigenstates in their own right. But this introduction is altogether too late: while its appearance in a course on relativistic quantum mechanics reflects accurately the *historical* development of the theory of antiparticles, it can tend to hide completely the fact that it is *relativistic mechanics itself* that makes possible the phenomenon of antiparticle motion—quantum mechanics is by no means a prerequisite.

Arguably, a thorough preliminary understanding of the *classical* theory of antiparticles better equips the student for tackling the same issues when they arise in relation to the Dirac equation. It is this topic that we shall review in this paper.

II. The proper time

Consider a structureless point particle. Classically, its kinematical state at any time t consists simply of the three components of its *three-position* $\boldsymbol{z}(t)$. We assume that $\boldsymbol{z}(t)$ is a continuous function of t that is sufficiently differentiable for our purposes. In special relativity, we form the *four-position* z^μ of the particle:

$$z^\mu \equiv (t, \boldsymbol{z}),$$

where we shall always use units in which $c = 1$. The continuous function $\boldsymbol{z}(t)$ specifies the *path* of the particle in Minkowski spacetime—its *worldline*. To parameterize its "length", in a Lorentz-invariant way, we consider an infinitesimal differential element of the path,

$$dz^{\mu}(t) \equiv (dt, d\boldsymbol{z}(t)),$$

where $d\boldsymbol{z}(t)$ is the infinitesimal change in position $\boldsymbol{z}(t)$ in the infinitesimal time interval from t to $t + dt$. We now consider the Lorentz-invariant quantity

$$d\tau^2(t) \equiv dz^{\mu}(t)dz_{\mu}(t) \equiv dt^2 - d\boldsymbol{z}^2(t), \tag{1}$$

where we employ a $(+,-,-,-)$ metric. What we would *like* to do is define a quantity $d\tau(t)$ that would provide a measure of "length" along the worldline. But the Lorentz-invariant expression (1) involves not $d\tau$, but rather the *square* of $d\tau$. Thus, $d\tau$ can only be defined *up to a sign*:

$$d\tau \equiv \pm\sqrt{dz^{\mu}dz_{\mu}}. \tag{2}$$

To investigate the meaning of this ambiguity in the sense of $d\tau$, let us consider the special case in which the particle is instantaneously at rest, with respect to our own inertial coordinate system:

$$d\boldsymbol{z} = \boldsymbol{0}.$$

In this case, we find

$$d\tau = \pm dt.$$

The solution $d\tau = dt$ for a particle at rest is the one usually presented in introductory texts on special relativity: such a $d\tau$ is obviously equal to the passage of time as measured in the *instantaneous rest frame* of the particle. For a particle undergoing arbitrary relativistic motion, we assume that the particle itself possesses its own "cumulative time" or "age", which we term the *proper time*, that can be calculated by summing up all of the $d\tau$ along its worldline:

$$\tau(\mathcal{E}) \equiv \int_{\mathcal{E}_0}^{\mathcal{E}} d\tau,$$

where $\tau(\mathcal{E})$ is the proper time at event $\mathcal{E}$ on the worldline, and where the event $\mathcal{E}_0$ on the worldline defines the (arbitrary) origin of τ. Since the worldline of any classical particle passes through each constant-t hyperplane once and only once, we can replace the events $\mathcal{E}$ and $\mathcal{E}_0$ by their corresponding coordinate times t and t_0, and hence determine τ as a function of t:

$$\tau(t) = \int_{t_0}^{t} dt' \, \frac{d\tau}{dt'} = \int_{t_0}^{t} dt' \, \frac{1}{\gamma(t')}, \tag{3}$$

where we have made use of Eq. (1):

$$\frac{d\tau(t)}{dt} = \sqrt{1 - \left(\frac{d\boldsymbol{z}(t)}{dt}\right)^2} \equiv \sqrt{1 - \boldsymbol{v}^2(t)} \equiv \frac{1}{\gamma(t)}.$$

The standard textbook result (3) shows that when the speed v of the particle is much smaller than the speed of light, the factor $\gamma(t)$ is close to unity, and the passage of proper time is indistinguishable from that of coordinate time; but if the particle's motion is such that its speed rises to an appreciable fraction of the speed of light, the factor $\gamma(t)$ rises above unity, and the particle "ages" more slowly. In all cases, however, the particle *gets older*: special relativity only seems to modify the rate; it warps our view of the world, but it does not throw it into reverse.

III. Classical antiparticles

Let us now consider the *other* solution in Eq. (2) for a particle at rest, namely,

$$d\tau = -dt. \tag{4}$$

Even to a student possessing a good knowledge of special relativity, Eq. (4) does not look familiar at all. It seems to imply that a particle at rest with respect to our Lorentz coordinate system might somehow believe that time evolves in the *opposite direction* to what we do! For example, if we determine that some spacetime event $\mathcal{E}'$ is definitely *earlier* than another event $\mathcal{E}''$ (i.e., $\mathcal{E}'$ lies within the backward lightcone of $\mathcal{E}''$), then a particle whose own "proper time" obeys (4) would insist, to the contrary, that $\mathcal{E}'$ is definitely *later* than $\mathcal{E}''$ (i.e., from the particle's point of view, $\mathcal{E}'$ lies within the *forward* lightcone of $\mathcal{E}''$).

The problem is that, by the principles of relativity, *such a particle is just as valid an observer of the universe as we are*: it agrees with us that the speed of light is unity in all inertial frames. We have no physically acceptable justification for dismissing its counter-intuitive view of the world. We must conclude that *both* of the solutions (2) are equally valid definitions of the passage of proper time. (This is analogous to the fact[4] that both the retarded *and* the advanced Liénard–Wiechert potentials for a point charge are equally valid solutions of Maxwell's equations.)

Armed with a thorough knowledge of relativistic quantum mechanics and quantum field theory, Stueckelberg[5] and Feynman[6,7,8] made the following realization: a particle for which $d\tau$ evolves in the opposite sense to the dt in our particular Lorentz frame of reference is simply in *antiparticle motion* with respect to us. Of course, there are no classical forces that can change "particle motion" into "antiparticle motion"—the two regimes are as disjoint as the interiors of the forward and backward lightcones; but, even classically, this does not bar the possibility that a particle might have *always* been in antiparticle motion.

Following this argument to its logical conclusion, it could be noted that there is a similar ambiguity of sign when parameterizing path lengths in *Euclidean* space, since there the invariant interval is also squared:

$$dl^2 \equiv d\boldsymbol{z}^2,$$

and hence we could equally well measure length one way along the path, or in the opposite way. But we are *already* used to the idea that, at a fundamental level, traveling to the left is no more difficult than traveling to the right. The crucial difference in Minkowski space is precisely the fact that classical forces *do not* reverse the sense in which "time is traversed"; our intuition with Galilean mechanics is rooted firmly in the belief that everyone agrees on the direction that time is traveling. Relativistically boosting to another frame of reference "warps" the rate at which clocks tick, but it does not reverse it; in contrast, time-reversal is a *discrete* symmetry, and cannot be brought into contact with "intuitive" physics by a continuous transformation.

IV. $\mathcal{C}$, $\mathcal{P}$, and $\mathcal{T}$

Another way of recognizing the possibility of the existence of antiparticle motion, in any relativistically complete theory of mechanics, is to consider the fundamental symmetries of the *Lorentz group*—namely, those transformations under which the interval (1) is invariant. Intuitive, introductory constructions of the proper time generally make use of everyday objects, such as people, trains, measuring rods, clocks, and so on. In thinking about such everyday objects—even in relativistic terms—we usually only consider *proper* Lorentz transformations (boosts and rotations)—or, at most, *orthochronous* ones (proper transformations with or without the parity transformation). But the interval (1) is also invariant under *non-orthochronous* Lorentz transformations—those involving the time-reversal operation—and it is precisely such transformations that convert what appears to be "normal" particle motion into "antiparticle" motion.

Let us make this argument more concrete. Consider the *parity* operation, which in classical physics simply changes the sign of the spatial coordinates in a given Lorentz frame:

$$\mathcal{P}:\ \boldsymbol{x} \to -\boldsymbol{x}.$$

The *time-reversal* operation does likewise for the time coordinate:

$$\mathcal{T}:\ t \to -t.$$

Under the combined operations of $\mathcal{P}$ and $\mathcal{T}$, all four components of dz^μ are reversed in sign:

$$\mathcal{P}\mathcal{T}:\ dz^\mu \to -dz^\mu. \tag{5}$$

The three-velocity, being a ratio of the spatial part $d\boldsymbol{z}$ to the temporal part dz^0, is therefore unchanged:

$$\mathcal{P}\mathcal{T}:\ \boldsymbol{v} \equiv \frac{d\boldsymbol{z}}{dt} \to \boldsymbol{v}.$$

Let us now *try* to define proper time so that the sign of $d\tau$ is always taken in the same sense as dt. Consider a free particle, that has a three-velocity $\boldsymbol{v}$ in a given Lorentz frame of reference; and let us define

$$d\tau = +\frac{dt}{\gamma}, \tag{6}$$

where we always define γ as the *positive* square-root:

$$\gamma \equiv \gamma(v) \equiv \frac{1}{\sqrt{1-v^2}}.$$

The choice of sign (6) lets us label the worldline with values of τ in the "standard" way, such that the τ values increase in the direction of increasing coordinate time t.

If we now apply the operation $\mathcal{PT}$ to the above Lorentz frame, we are placed in a new, equally valid Lorentz frame, in which all directions—space *and* time—have been reversed. But this operation does not affect our τ markings on the particle's worldline, since the proper time is a property of the particle itself. Thus, with respect to the *new* Lorentz frame, τ increases in the direction of *decreasing* coordinate time t:

$$d\tau = -\frac{dt}{\gamma}.$$

Thus, if we insist on the invariance of classical mechanics under the complete Lorentz group (as we do in all other forms of relativistic mechanics), we find that for every possible solution of the equations of motion with the choice of the *positive* sign in Eq. (2), there exists an equally possible solution in which the *negative* sign is chosen. By the Stueckelberg–Feynman interpretation, this transformation is the classical *particle–antiparticle* (or "charge-conjugation") transformation, so let us label it as such:

$$\mathcal{C}: \ \tau \to -\tau. \tag{7}$$

Let us denote by β the choice of sign in Eq. (2): $\beta = +1$ if the particle is in "normal particle" motion with respect to our own Lorentz frame (i.e., $d\tau/dt > 0$); whereas $\beta = -1$ if the particle is in "antiparticle" motion (i.e., $d\tau/dt < 0$). (The symbol β is sometimes used in introductory texts for the ratio v/c, but in natural units it is simpler to just use the intuitive symbol v for this latter quantity. It is to maintain consistency with the results of the Foldy–Wouthuysen transformation[1,3] that we use the symbol β for the classical particle–antiparticle number.)

With the definition (7), classical mechanics possesses all three discrete symmetry operations $\mathcal{C}$, $\mathcal{P}$, and $\mathcal{T}$ required for a relativistically invariant system of mechanics. Classically, all of these operations commute, and each of them individually squares to unity. We expect that the equations of motion of classical physics will be invariant under the combined operation $\mathcal{CPT}$.

V. Antiparticles in the real world

Let us now show that the classical $\mathcal{C}$ operation yields "antiparticles" in the everyday sense of the word, i.e., that the antiparticle of an electron is a positron, and so on. To do so, it suffices to give our classical point particle two characteristics: an electric charge q, and a mass m. We assume that the quantities q and m are Lorentz scalars, and are *unchanged* under any of the operations $\mathcal{C}$, $\mathcal{P}$, or $\mathcal{T}$, as defined above. (We shall show how the usual interpretation of $\mathcal{C}$ as changing the sign of the "effective" charge is to be understood shortly.)

Let us first consider the *electromagnetic* interaction. This is a *vector* interaction, which couples to the *electromagnetic vector current density* of the point charge,[4]

$$j^\mu(x) \equiv q\int_{-\infty}^{\infty} d\tau\, \delta^{(4)}[x - z(\tau)]\, u^\mu(\tau), \tag{8}$$

where $u^\mu(\tau)$ is the *four-velocity* of the particle:

$$u^\mu \equiv \frac{dz^\mu}{d\tau} = (\beta\gamma, \beta\gamma\boldsymbol{v}). \tag{9}$$

The delta function and τ-integration in (8) are necessary to obtain a current *density* from the trajectory of the point particle (the density of a point particle being infinite on its worldline and zero outside); but the essential properties of $j^\mu(x)$ are contained in the classical *electromagnetic current vector*,

$$J^\mu \equiv q u^\mu. \tag{10}$$

From Eq. (9) we see that u^μ is both $\mathcal{C}$-odd and $\mathcal{PT}$-odd, since dz^μ is $\mathcal{C}$-even and $\mathcal{PT}$-odd, whereas $d\tau$ is by definition $\mathcal{C}$-odd and $\mathcal{PT}$-even. Thus, since q is assumed to be unchanged by $\mathcal{C}$ or $\mathcal{PT}$ as we have defined them, J^μ is also $\mathcal{C}$-odd and $\mathcal{PT}$-odd. (The density $j^\mu(x)$ clearly possesses the same symmetries as J^μ, since the delta function in (8) is effectively an even function of its argument, and under the $\mathcal{C}$ operation the sign of $d\tau$ is reversed, but so too are the limits of integration.) The former property is of particular importance for us:

$$\mathcal{C}: \; J^\mu \to -J^\mu. \tag{11}$$

The result (11) tells us that, as far the electromagnetic interaction is concerned, antiparticle ($\beta = -1$) motion of a charged particle *appears the same* as an "equivalent normal particle", with $\beta = +1$, with the same three-velocity $\boldsymbol{v}$ as the original particle, but with the *opposite "effective" charge*, since

$$J^\mu \equiv q\,\frac{dz^\mu}{d\tau} = (-q)\,\frac{dz^\mu}{d(-\tau)}.$$

For example, a particle of charge q, at rest, but in antiparticle motion, generates a static Coulomb field that is equivalent to that from a "normal" particle at rest of charge $-q$. It is in this sense—of replacing antiparticle motion by an "equivalent normal particle" with an opposite "effective" charge—that the classical particle–antiparticle operation $\mathcal{C}$ is a "charge conjugation" operation.

Let us now determine the *mass* of the "equivalent normal particle" corresponding to antiparticle motion. A priori, it may not be clear how we can make such a determination. However, if we believe Einstein's theory of general relativity (which we do here), then the equivalence principle tells us that "inertial" mass and "gravitational" mass are one and the same thing: they both represent the coupling of *matter* to *spacetime*. Thus, we can simply carry out the same analysis as we performed above for the electromagnetic interaction, but now with regard to the *gravitational* interaction.

Gravitation is a *tensor* interaction, which couples to the *mechanical stress–energy tensor current density* of our point particle:

$$t^{\mu\nu}(x) \equiv m\int_{-\infty}^{\infty} d\tau\, \delta^{(4)}[x - z(\tau)]\, u^\mu(\tau)u^\nu(\tau). \tag{12}$$

Again, the delta function and integration over τ and are required for the purposes of converting a pointlike trajectory into a *density*; the essential properties of $t^{\mu\nu}(x)$ are contained in the classical *mechanical current tensor*,

$$T^{\mu\nu} \equiv m u^\mu u^\nu. \tag{13}$$

In this case, we find that $T^{\mu\nu}$ is $\mathcal{C}$-*even*, since it contains *two* factors of u^μ:

$$\mathcal{C}:\ T^{\mu\nu} \to T^{\mu\nu}.$$

Thus, as far as the gravitational interaction is concerned, a particle of mass m in antiparticle motion ($\beta = -1$) is indistinguishable from the same particle of mass m in the corresponding particle motion ($\beta = +1$). For example, the gravitational field of a star made of antimatter is the *same* as that of an identical star in which the antimatter is replaced by normal matter; a collection of such stars would all *attract* each other gravitationally. By the equivalence principle, this invariance of the mass of the "equivalent normal particle" under the $\mathcal{C}$ operation is true in full generality.

The above examples have concentrated on the fields *generated* by particles in antiparticle motion, but the same conclusions can be drawn from the equations of motion for the particles *under the influence* of given external fields. In the gravitational case, the mass m actually drops out of the equations of motion (again, by the equivalence principle), and the equation of motion is simply the *geodesic equation*,[9]

$$\frac{du^\mu}{d\tau} = -\Gamma^\mu{}_{\nu\rho} u^\nu u^\rho. \tag{14}$$

If the particle is in antiparticle motion, we can again replace it by an "equivalent normal particle": since

$$\frac{du^\mu}{d\tau} \equiv \frac{d^2 z^\mu}{d\tau^2} = \frac{d^2 z^\mu}{d(-\tau)^2} \tag{15}$$

and

$$u^\mu \equiv \frac{dz^\mu}{d\tau} = -\frac{dz^\mu}{d(-\tau)}, \tag{16}$$

we can rewrite the geodesic equation (14) in the form

$$\frac{d^2 z^\mu}{d(-\tau)^2} = -\Gamma^\mu{}_{\nu\rho} \frac{dz^\nu}{d(-\tau)} \frac{dz^\rho}{d(-\tau)},$$

which shows that the "equivalent normal particle" acts in the same way as any other particle.

In the electromagnetic case, the equation of motion is the *Lorentz force law*, which in relativistic form is

$$\frac{du^\mu}{d\tau} = \frac{q}{m} F^{\mu\nu} u_\nu, \tag{17}$$

where $F^{\mu\nu}$ represents an external electromagnetic field. Again, if the particle is in antiparticle motion, we can use the properties (15) and (16) to rewrite Eq. (17) in the form

$$\frac{d^2 z^\mu}{d(-\tau)^2} = -\frac{q}{m} F^{\mu\nu} \frac{dz_\nu}{d(-\tau)}.$$

Hence the "equivalent normal particle" has the sign of its effective *charge-to-mass ratio* negated compared to that of the actual particle; and since we have already determined that its *mass* is unchanged, we conclude that it is its "effective charge" that changes sign, in agreement with our analysis above.

VI. The "negative energy problem"

Finally, let us discuss a subtlety that is a frequent source of confusion for the student: the existence and interpretation of an *energy–momentum four-vector* for antiparticles. The subject arises most naturally when we consider the Lagrangian and Hamiltonian formulations of mechanics (which are of course of vital importance in the construction of a quantum mechanical description); but in introductory courses and textbooks it is often presented in a somewhat confusing and contradictory manner.

Let us continue to consider our classical point particle of mass m and charge q. Relativistically, the construction of a manifestly covariant set of generalized coordinates is somewhat delicate:[4,10] we would like to treat *all four* components z^μ of the position four-vector of the particle in an equal fashion, with the "time" parameter preferentially given by the Lorentz-invariant proper time τ. But the existence of the definition (1) tells us that only *three* of the components of z^μ are actually independent of τ; and the Lagrangian and Hamiltonian formulations of mechanics are greatly complicated if all of the generalized coordinates are not actually independent.[10]

One way to circumvent these problems is to take the "time" parameter of the Lagrangian or Hamiltonian formalism to be the "θ-time", where

$$d\theta \equiv \frac{d\tau}{m}. \tag{18}$$

The quantity θ is a Lorentz scalar, since both τ and m are Lorentz scalars; and since m is even under $\mathcal{C}$, $\mathcal{P}$, and $\mathcal{T}$, the quantity θ has the same symmetry properties as τ; in particular,

$$\mathcal{C}: \ \theta \to -\theta.$$

To show that the simple redefinition (18) solves the independence problem, one need simply note that the *mass* (rest-energy) $m(\tau)$ of a *general* system need not be a constant of the motion. Thus, while one of the four components of z^μ is still differentially dependent on τ through Eq. (1), *all four* components of z^μ are, in general, independent of θ, since the m in Eq. (18) may vary as a function of τ. A useful bonus of this approach is that *all* equations in the θ-time formalism can be written in such a way that that the quantities m and τ never explicitly appear; the formalism may then be applied equally well to *massless* particles (for which θ exists, but for which $m = 0$ and τ is undefinable).

A somewhat simplified version of this approach is not to actually use the θ-time at all, but rather to simply "pretend" that τ is not in fact constrained by Eq. (1) until *after* the equations of motion have been obtained. The results are the same, so let us follow this latter, simplified approach. A suitable Lagrangian can, for example, be chosen to be[5]

$$L = \frac{1}{2} m u^\mu u_\mu + q u^\mu A_\mu, \tag{19}$$

where A_μ is the electromagnetic four-potential:

$$A^\mu \equiv (\varphi, \boldsymbol{A}).$$

Using Eqs. (10) and (13), we can recognize the Lagrangian (19) as simply a straightforward coupling of the tensor and vector currents of the particle to their corresponding gauge fields:

$$L = \frac{1}{2} T^{\mu\nu} g_{\mu\nu} + J^{\mu} A_{\mu}.$$

The canonical momentum components p_μ conjugate to the generalized degrees of freedom z^μ are obtained from (19) in the standard way:

$$p_\mu \equiv \frac{\partial L}{\partial u^\mu} = m u_\mu + q A_\mu. \tag{20}$$

The Euler–Lagrange equations of motion then yield the Lorentz force law (17).[11]

A corresponding *Hamiltonian* formulation of this same classical theory may be constructed in two different ways. On the one hand, a "manifestly-covariant Hamiltonian" $\mathcal{H}$ can be constructed by the usual Legendre transformation:

$$\mathcal{H} \equiv p_\mu u^\mu - L = \frac{1}{2m}(p^\mu - qA^\mu)(p_\mu - qA_\mu) \equiv \frac{(p - qA)^2}{2m},$$

from which Hamilton's equations (with respect to the "time" parameter τ) again yield the Lorentz force law (17), as well as the equation (20) relating the canonical momentum and velocity four-vectors.

On the other hand, the more *conventional* way to construct a Hamiltonian formulation of this system—that yields a somewhat simpler transition to the quantum theory—is to recognize that the *canonical energy* p^0 can be interpreted as the Hamiltonian H of the system, with respect to the coordinate time t. From Eq. (20), we have

$$(p - qA)^2 = m^2 (u^\mu u_\mu) \equiv m^2,$$

where in this last expression we can "stop pretending" that $u^\mu u_\mu$ is not identically equal to unity, because it is p^μ, not u^μ, that plays a fundamental role in the Hamiltonian formulation of mechanics. Thus

$$(H - q\varphi)^2 - (\boldsymbol{p} - q\boldsymbol{A})^2 = m^2,$$

and hence

$$H = q\varphi + \beta\sqrt{m^2 + (\boldsymbol{p} - q\boldsymbol{A})^2}, \tag{21}$$

where $\beta = \pm 1$ encapsulates the choice of sign in taking the square root. Hamilton's equations again yield expressions equivalent to (17) and (20).

Let us now consider the simple case of a free particle, which will be sufficient for our purposes. From Eq. (20), we have in this case

$$p^\mu = m u^\mu; \tag{22}$$

equivalently, from Eq. (21), we have

$$H = \beta\sqrt{m^2 + \boldsymbol{p}^2}. \tag{23}$$

Eqs. (22) and (23) emphasize the fact that, in antiparticle motion, the canonical momentum four-vector p^μ has *negative energy*; or, in other words, that p^μ is *odd* under $\mathcal{C}$:

$$\mathcal{C}:\ p^\mu \to -p^\mu.$$

Consider, now, a particle and its corresponding antiparticle, both at rest. For the former, we have

$$p^\mu = (m, \mathbf{0}), \tag{24}$$

whereas for the latter we have

$$p^\mu = (-m, \mathbf{0}). \tag{25}$$

"Surely," a common argument goes, "does this not tell us that antiparticle motion is really a *negative mass* solution? Does it not further tell us that the *total* energy of this pair of particles is *zero*?" These two statements directly contradict our finding above that the mass of a particle is *unchanged*, whether it be in particle or antiparticle motion; and we of course know that the total energy of a neutral particle–antiparticle pair at rest is indeed $2m$, *not* zero. On the other hand, we know that, quantum mechanically, the canonical energy *really is* negative for an antiparticle solution: for example, the eigenstates for $\boldsymbol{p} = \mathbf{0}$ have a time-dependence of the form $e^{\mp imt}$ (where we use units in which $\hbar = 1$), which, using the Einstein relation $E = i\partial/\partial t$, implies that $E = \pm m$. So what are we to believe?

This "paradox" is usually presented in discussions of relativistic quantum mechanics—leading to Dirac's "holes", and so on—but it is fundamentally a feature of relativistic mechanics itself, whether it be of the classical *or* quantum flavor. Let us now dispense with it once and for all.

The fallacy above is the assumption that the canonical momentum four-vector has anything at all to do with the "total mass" of a system. It does not. In trying to "add together" the four-momenta of the two particles, we are making the implicit assumption that we are in some sense computing the four-momentum generalization of the "total energy" or "total rest mass" of a system. But we have already noted that for a system of *gravitational* sources, it is the sum of the *mechanical stress-energy tensor densities* $t^{\mu\nu}(x)$ that determines the overall gravitational field generated by the system. From this gravitational field, one can define a "total gravitational mass" of the system (because for our purposes the strengths of these gravitational fields are assumed to be negligible compared to the other forces present, so that special relativity is a good approximation). But by the equivalence principle, this "gravitational" mass is simply *the* mass of the system; and *no definition of "mass" giving a different result can be compatible with the equivalence principle*. Now, we know that $T^{\mu\nu}$ is *unchanged* under the $\mathcal{C}$-operation; in particular, for both the particle *and* the antiparticle above, we have

$$T^{00} = m,$$
$$T^{0i} = T^{i0} = T^{ij} = 0,$$

and so for the system as a whole we have $T^{00} = 2m$. Thus, the total mass of the particle–antiparticle pair at rest *is* indeed the commonsense result of $2m$, not zero.

Let us review these issues in more detail. The Hamiltonian H is the zero-component of the canonical momentum four-vector p^μ, and is thus reasonably called the "canonical energy". In quantum mechanics, the canonical momentum four-vector relates directly to frequency and wavelength:

$$p_\mu \to i\partial_\mu.$$

Thus, particle (antiparticle) motion for a free particle corresponds to positive (negative) frequencies. But this has nothing at all to do with those *mechanical* (or "kinematical") properties of the particle, that are physically observable. Indeed, in the presence of interactions, even the *sign* of the canonical energy (frequency) loses its relevance completely: for example, for the electromagnetic interaction of a classical point charge, we have

$$p^\mu = mu^\mu + qA^\mu.$$

It will be immediately noted that p^μ is not gauge-invariant, and the value of p^0 can be given any arbitrary value (positive *or* negative) simply by redefining the zero point of the scalar potential. Thus p^μ cannot possibly, of itself, determine any physically observable property of the particle, such as its mass, or its mechanical stress–energy tensor.

On the other hand, we know that it *is* possible to define some sort of four-vector π^μ that represents the *mechanical* energy–momentum of a system—after all, we have been adding energies and momenta together for centuries. To define such a π^μ, we need simply integrate the *mechanical* stress-energy tensor density $t^{\mu\nu}(x)$ over all three-space, in some *given* Lorentz frame. We can write this covariantly in the form

$$\pi^\mu(t) \equiv \int d^3\sigma_\nu \, t^{\mu\nu}(t, \boldsymbol{x}), \tag{26}$$

where an element of "three-space" has been written covariantly as $d^3\sigma^\mu$: in the given Lorentz frame, in which it is simply all three-space at a constant time t, we have

$$d^3\sigma^\mu = n^\mu \, d^3x,$$

where n^μ is the timelike *four-normal* to the hyperplane, which in this frame has coordinates

$$n^\mu = (1, \mathbf{0}).$$

Let us compute π^μ in this Lorentz frame: from Eqs. (12) and (26), we have

$$\pi^\mu(\tau) \equiv m \int d^3x \int_{-\infty}^{\infty} d\tau \, \delta^{(4)}[x - z(\tau)] \, u^\mu(\tau) u^0(\tau),$$

where we write $\pi^\mu(\tau)$ on the understanding that the given τ is the proper time of the particle at the corresponding coordinate time t. The integration over all three-space can be performed immediately, with the three-part of the delta function simply yielding unity:

$$\pi^\mu(\tau) = m \int_{-\infty}^{\infty} d\tau \, \delta[t - z^0(\tau)] \, u^\mu(\tau) u^0(\tau).$$

We can now perform the integration over τ by noting the standard identity for a delta function of a function:

$$\delta[f(\tau)] = \sum_{\tau_z} \frac{\delta(\tau - \tau_z)}{|df/d\tau(\tau_z)|},$$

where τ_z are the zeroes of $f(\tau)$. In this case we have

$$f(\tau) = t - z^0(\tau),$$

whence

$$\left|\frac{df}{d\tau}\right| = |u^0|,$$

and hence

$$\pi^\mu = m u^\mu \frac{u^0}{|u^0|},$$

where all quantities are assumed evaluated at the given value of τ (or t). We now recognize the last factor as the *particle–antiparticle number* β:

$$\beta \equiv \frac{u^0}{|u^0|} = \pm 1,$$

and hence

$$\pi^\mu = \beta m u^\mu = (m\gamma, m\gamma \boldsymbol{v}). \tag{27}$$

We see that the extra factor of β "cancels" the oddness of u^μ under $\mathcal{C}$, so that the four-vector π^μ is—like $T^{\mu\nu}$ itself—*even* under the $\mathcal{C}$ operation. *It is the mechanical momenta* π^μ*, not the canonical momenta* p^μ*, that should be added together to compute the mechanical four-momentum of a system of particles.* (Similarly, one must be careful to never confuse the *mechanical* stress–energy tensor density $t^{\mu\nu}(x)$ with the *canonical* stress–energy tensor density $\Theta^{\mu\nu}(x)$ of field theory:[12] the former can be derived by functional differentiation of the Lagrangian with respect to the *metric tensor* $g_{\mu\nu}$, and is therefore always symmetric;[9,13] the latter is defined in field theory with respect to the canonical momentum density, and in general possesses no particular symmetry.)

We can perform a similar integration of the *electromagnetic* current density vector $j^\mu(x)$, to obtain the "effective flux of charge" Q passing through a given spacelike hypersurface:

$$Q(t) \equiv \int d^3\sigma_\mu \, j^\mu(t, \boldsymbol{x}).$$

By a similar analysis to that above, we simply find

$$Q = \beta q,$$

so that Q can be understood as the "effective" charge of the particle, if it had been in "normal" particle motion. (The choice of what is "particle" motion and what is "antiparticle" motion is of course implicitly contained in the *direction* of the timelike four-normal n^μ.)

Quantity	Symbol	Rank	$\mathcal{C}$	$\mathcal{PT}$	Value
Mass	m	0	+	+	Free parameter
Electric charge	q	0	+	+	Free parameter
Proper time	τ	0	−	+	$d\tau = \beta\, dt/\gamma(t)$
Theta time	θ	0	−	+	$d\theta = d\tau/m(\tau)$
Particle number	β	0	−	+	$u^0/\lvert u^0\rvert = u^0/\gamma = \pm 1$
Lagrangian	L	0	+	+	$T^{\mu\nu}g_{\mu\nu}/2 + J^\mu A_\mu$
Hamiltonian	$\mathcal{H}$	0	+	+	$p_\mu u^\mu - L = (p - qA)^2/2m$
Effective charge	Q	0	−	+	βq
Position	z^μ	1	+	−	$z^\mu(\tau)$, state vector
Velocity	u^μ	1	−	−	$dz^\mu/d\tau = (\beta\gamma, \beta\gamma\boldsymbol{v})$
Canonical momentum	p^μ	1	−	−	$\partial L/\partial u_\mu = mu^\mu + qA^\mu$
Electromagnetic potential	A^μ	1	−	−	$\partial_\mu\partial^\mu A_\nu - \partial_\mu\partial_\nu A^\mu = j_\nu$
Electromagnetic current	J^μ	1	−	−	qu^μ
Mechanical momentum	π^μ	1	+	−	$\beta m u^\mu = (m\gamma, m\gamma\boldsymbol{v})$
Electromagnetic field	$F^{\mu\nu}$	2	−	+	$\partial^\mu A^\nu - \partial^\nu A^\mu$
Mechanical stress–energy	$T^{\mu\nu}$	2	+	+	$mu^\mu u^\nu$
Mechanical angular momentum	$M^{\mu\nu\rho}$	3	+	−	$z^\nu T^{\mu\rho} - z^\rho T^{\mu\nu}$

Table 1: Lorentz-covariant quantities for a classical point charge.

VII. Conclusions

The Stueckelberg–Feynman picture of antiparticles being simply particles “moving backwards in proper time” can be seen to be an integral and important part of relativistic classical mechanics, which only requires minor additions to standard texts on special relativity. Furthermore, the historical misconceptions of the “negative energy problem” in relativistic quantum mechanics can be avoided by a thorough understanding of the difference between the *canonical* momentum p^μ of Lagrangian theory, and the “mechanical” momentum π^μ that dictates the kinematical and gravitational properties of an object.

Finally, we summarize the properties of the various important Lorentz-covariant quantities for a classical point charge in Table 1.

Acknowledgments

Helpful discussions with S. A. Wouthuysen, J. W. G. Wignall, A. G. Klein, G. I. Opat, and M. J. Thomson are gratefully acknowledged. This work was supported in part by the Australian Research Council. We were saddened to learn of the death of Prof. Wouthuysen during the completion of this work.

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