Insert to page 250: Proof of Eq. (6.37)

One of the Examiners of this thesis, Professor Gerard J. Stephenson Jr., has supplied an analytical proof of the integral (6.37), and has given permission for it to be included in this insert and attributed to him. I reproduce here the details, as they appear in his Examiner's Report (a few words are added to describe his diagram, and the visual layout of the proof).

Define $\xi \equiv \cos\angle xy$, where $\angle xy$ is the angle between two three-vector variables $\boldsymbol{x}$ and $\boldsymbol{y}$. We then have

$$\int\limits_{|x|<R} d^3x \int\limits_{|y|<R} d^3y \, \frac{(\boldsymbol{x}+\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{y})^2} = 2\pi \int\limits_{|x|<R} d^3x \int_0^R dy\, y^2 \int_{-1}^1 d\xi \, \frac{x^2+y^2+2xy\xi}{x^2+y^2-2xy\xi}$$

$$= 8\pi^2 \int_0^R dx\, x^2 \int_0^R dy\, y^2 \left\{ \frac{x^2+y^2}{xy} \ln \frac{x+y}{x-y} - 2 \right\}.$$

The apparent singularity at $x = y$ is benign, since

$$\lim_{y \to x} (x-y) \ln(x-y) = 0.$$

To be careful, do $\int_0^R dy$ as $\int_0^x dy + \int_x^R dy$; most of the terms go out or cancel. There are several remaining terms; *see*, *e.g.*, Gradshteyn and Ryzhik 2.729.1. The log term in the integral above gives

$$5\left(\frac{4\pi}{3}\right)^2 R^6;$$

the -2 term gives

$$-\left(\frac{4\pi}{3}\right)^2 R^6;$$

after multiplying the integral above by

$$\left(\frac{4\pi R^3}{3}\right)^{-2},$$

the net result is

$$\left(\frac{4\pi R^3}{3}\right)^{-2} \left\{ 5\left(\frac{4\pi}{3}\right)^2 R^6 - \left(\frac{4\pi}{3}\right)^2 R^6 \right\} = 4,$$

in agreement with Eq. (6.37).