UM–P–95/12
RCHEP–95/08

The Foldy–Wouthuysen transformation*

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(31 January 1995)

Abstract

The Foldy–Wouthuysen transformation of the Dirac Hamiltonian is generally taught as simply a mathematical trick that allows one to obtain a two-component theory in the low-energy limit. It is not often emphasised that the transformed representation is the only one in which one can take a meaningful *classical limit*, in terms of particles and antiparticles. We briefly review the history and physics of this transformation.

1. Introduction

In relativistic quantum mechanics, one often identifies those components of the wavefunction that represent "particles", and those that represent "antiparticles". But there always exist *canonical transformations* of the wavefunction (changes of representation) that mix these particle and antiparticle components together, while still leaving the physical quantities represented by the theory unchanged, as long as the operators are complementarily transformed. This means that the components of the wavefunction that appear to represent antiparticles in one representation will actually be a *superposition* of particle and antiparticle components in a different representation.

It would be difficult to recognise a classical limit of the relativistic quantum theory if this arbitrariness in representation were to be permitted to run free. Classical physics does not have any trouble with the concept of antiparticles *per se*: by Feynman's interpretation, antiparticle motion is simply effected by means of the "classical C" transformation

$$\tau \to -\tau$$

on the corresponding particle motion, where τ is the proper-time of the particle. But the discreteness of this classical C transformation—and the lack of any sort of "superposition" principle—mean that classical physics does *not* admit any "mixing" of particle and antiparticle motion.

2. Newton and Wigner

How, then, can one obtain a meaningful classical limit of relativistic quantum mechanics? The clue to the path out of this dilemma was first found in 1949 by Newton and Wigner [2], as almost a by-product of other, more abstract considerations. The findings of Newton and Wigner eradicated some of the myths surrounding the *position operator* in relativistic wave equations—in particular, that states localised in position cannot be formed solely from positive-energy states; and that if a particle's position is measured below its Compton wavelength, one necessarily generates particle–antiparticle pairs, which renders the position measurement of a single particle impossible. In pursuing some rather simple questions of a group theoretical nature, they not only found what they were looking for, but also some unexpected bonuses. These were explained and elaborated on by Foldy and Wouthuysen [3], who also obtained the explicit transformation that realised the goals of Newton and Wigner for the physically important case of a spin-half particle. (Case [4] later generalised their method to spin-zero and spin-one particles.)

The original aim of Newton and Wigner was to rigorously formulate the properties of *localised states*, for arbitrary-spin relativistic representations of elementary particles. They proceeded simply on the basis of *invariance requirements*. They sought a set of states which were localised at a certain point in space, such that any state becomes, after a translation, orthogonal to all of the undisplaced states; such that the superposition of any two such localised states is again a localised state in the set; that the set of states be invariant under rotations about the point of localisation, and under temporal and spatial reflections; and that the states all satisfy certain regularity conditions, amounting to the requirement that all of the operators of the Lorentz group be applicable to them.

From such a simple and reasonable set of requirements, a most bountiful crop was harvested. Firstly, Newton and Wigner found that the set of states they sought *could*, indeed, be found, for arbitrary spin (provided the mass is non-zero); moreover, their requirements in fact specify a *unique* set of states with the desired properties. Furthermore, these states are all *purely positive-energy states* (or, equivalently, purely negative-energy). They further belong to a *continuous eigenvalue spectrum of a particular operator*, which itself has the property of preserving the positive-energy nature of the wavefunction.

Due to these remarkably agreeable properties, Newton and Wigner felt that one would be justified in referring to the operator they had found as *the* position operator—in contradistinction to the operator $\boldsymbol{x}$ in some arbitrary representation of the relativistic wave equation, which only is the "position" operator *in that particular representation*, and hence has no invariant physical meaning, since the representation may be subject to an (in general position-dependent) canonical transformation, that by definition cannot change any physical quantities, but which most definitely changes the expectation values of the fixed operator $\boldsymbol{x}$. The Newton–Wigner position operator had, in fact, been discovered previously in 1935 by Pryce [5], who found the operator a useful tool in the Born–Infeld theory, and, later [6], in a discussion of relativistic definitions of the centre of mass for systems of particles.

3. Foldy and Wouthuysen

A natural question to ask, given the findings of Newton and Wigner, is the following: What does a given relativistic wave equation look like in the representation in which the Newton–Wigner position operator *is*, in fact, simply the vector $\boldsymbol{x}$? This is the question effectively asked by Foldy and Wouthuysen in their classic 1950 paper [3], for the physically important case of the Dirac equation. (Their stated aim was actually to find a representation in which the components for positive- and negative-energy states are decoupled, but from the above it is clear that this is effectively the same as seeking the Newton–Wigner representation.) What they found is, even today, simply astounding. Firstly, they found that the canonical transformation from the Dirac–Pauli representation to the Newton–Wigner representation of the *free* Dirac equation is, in fact, obtainable exactly. Secondly, they found that the *Hamiltonian* for the free particle, in the Newton–Wigner representation, agrees completely with that of classical physics,

$$H_{\mathrm{NW}} = \beta(m^2 + \boldsymbol{p}^2)^{1/2} \equiv \beta W_p \tag{1}$$

(we use units in which $\hbar = c = 1$), in contrast to that of the Dirac–Pauli representation,

$$H_{\mathrm{DP}} = \beta m + \boldsymbol{\alpha}\cdot\boldsymbol{p},$$

which—while having the important property of linearity—does not resemble the classical expression at all.

Thirdly, Foldy and Wouthuysen found that the *velocity operator* (obtained from the position operator by means of its Heisenberg equation of motion) in the Newton–Wigner representation—or, equivalently, the corresponding Newton–Wigner velocity operator in *any* representation—satisfies the *classical* relation for a free particle:

$$\boldsymbol{v}_{\mathrm{NW}} \equiv \frac{d}{dt}\boldsymbol{x}_{\mathrm{NW}} = \beta\frac{\boldsymbol{p}}{W_p}. \tag{2}$$

That (2) is an amazing result is recognised from the fact that, from the very inception of the Dirac equation, it was known that the "velocity" operator in the *Dirac–Pauli* representation *does not* make any classical sense whatsoever: its sole eigenvalues are plus or minus the speed of light; it is not directly related to the momentum $\boldsymbol{p}$; and its equation of motion has non-real "*zitterbewegung*" oscillatory motion (*see*, *e.g.*, [7]). In drastic contradistinction, the Newton–Wigner velocity operator $\boldsymbol{v}_{\mathrm{NW}}$ of (2) has the physically understandable continuum of eigenvalues between plus and minus the speed of light; its relationship to the momentum of the free particle is identical to that valid in classical physics; and, when one considers in turn *its* Heisenberg equation of motion, then one finds that, for a free particle, the velocity $\boldsymbol{v}_{\mathrm{NW}}$ is a constant, since $\boldsymbol{p}$ and W are also.

Fourthly, Foldy and Wouthuysen found that the free-particle spin and orbital angular momentum operators in the Newton–Wigner representation—defined to be simply $\boldsymbol{l}_{\mathrm{NW}} \equiv \boldsymbol{x}\times\boldsymbol{p}$ and $\boldsymbol{\sigma}_{\mathrm{NW}} \equiv \boldsymbol{\sigma}$ in this representation—are *constants of the motion separately*; again, it is well-known [7] that, in the Dirac–Pauli representation, these operators are *not* separately constants of the motion, even for a free particle. (The peculiarity of the Dirac–Pauli representation in this respect can be traced back to the fact that the "position" operator in that representation exhibits the non-physical "*zitterbewegung*" motion, which thus enters into the motion of the "orbital angular momentum" operator $\boldsymbol{x}_{\mathrm{DP}}\times\boldsymbol{p}$ in this representation.)

As a fifth and final accomplishment, Foldy and Wouthuysen attacked the problem of finding the canonical transformation from the Dirac–Pauli representation to the Newton–Wigner representation, in the case of the *electromagnetically-coupled* Dirac equation. Unfortunately, this cannot be done in closed form. Nevertheless, Foldy and Wouthuysen showed how one can obtain successive approximations to the required transformation, as a power series in

p^α/m, qA^α/m, for an arbitrary initial Hamiltonian H_{DP} in the Dirac–Pauli representation; this is the transformation that is presented in almost any textbook on relativistic quantum mechanics (*see*, *e.g.*, [8]).

An unstated assumption, crucial to the validity of the power series implementation of the transformation, is that the "odd" part of the Hamiltonian is in fact of no higher order in m than m^0. This is usually the case, but the assumption has the latent ability to trip one up. For example, if one tries to transform a Hamiltonian in which the mass term βm has been multiplied by $e^{i\theta\gamma_5}$—say, by a canonical transformation of the representation,—then one can be led to quite erroneous conclusions if one assumes that the terms omitted in the subsequent power series Foldy–Wouthuysen expansion are of high order in p^α/m and qA^α/m; in fact, the omitted terms are of exactly the same order as the terms that are retained; the power series method is, if applied in this way, completely useless. In such cases, the correct procedure is to first perform a simple canonical transformation to remove the order m^{+1} terms from the "odd" parts of the Hamiltonian; the resulting representation may then be fruitfully subjected to the power series Foldy–Wouthuysen transformation.

4. The Dirac–Pauli representation

It may be wondered, after hearing of all of the wonderful properties of the Newton–Wigner representation, why one should bother with any other representation at all. In particular, why do we usually only concentrate on the Dirac–Pauli representation of the Dirac equation? (Or representations "trivially" related to it; we shall define this term with more precision shortly.) The answer is subtle, but beautiful. *The charged leptons in Nature are well described by a minimal coupling of their Dirac fields to the electromagnetic field, in the Dirac–Pauli representation only.* It is not often stressed that *minimal coupling*—the use of the prescription

$$p \to p - qA$$

in the corresponding non-interacting formalism—is *not* a universal, representation-independent transformation. The reason is that, in general, a canonical transformation used to effect a change in representation may be *momentum-dependent*; indeed, the Foldy–Wouthuysen transformation itself is an important example. Clearly, the processes of using minimal coupling, and then performing a momentum-dependent transformation, on the one hand; and that of performing the momentum-dependent transformation first, and *then* using minimal coupling, on the other; will lead to completely *different* relativistic wave equations, in general. *A priori, one cannot know which representation one should use the minimal coupling prescription on.*

(Clearly, "trivial" changes of representation, in the sense used above, are therefore those in which the canonical transformation does not involve the momentum operator.)

We thus see that, by his insistence on a *linear* relationship between $\boldsymbol{p}$ and H—for reasons that were rendered obsolete by second quantisation—Dirac was led to the one representation of the spin-half Hamiltonian in which the assumption of minimal coupling gives the correct electromagnetic interactions for the electron, and in particular the correct gyromagnetic ratio and hydrogen spectrum. With hindsight, we can see that Dirac was both brilliant and lucky.

5. The two faces of the electron

We therefore come to recognise that there are *two* representations of the Dirac equation that are singled out above all others—each having qualities unique to itself—that have a truly direct correspondence with Nature: The Dirac–Pauli representation is unique due to its linearity; it is the representation in which the charged leptons are minimally coupled. The Newton–Wigner representation is unique due to its decoupling of positive- and negative-energy states; it is the representation in which the operators of the theory correspond to their classical counterparts.

We may go even further, conceptually speaking, in our description of the charged leptons: they are, in effect, two types of particle in the one being. On the one hand, in the Dirac–Pauli representation, all four components are inextricably coupled, but the particles are *pure, pointlike, structureless electric charges.* On the other hand, in the Newton–Wigner representation, operators act quite in accord with classical mechanics, but the electromagnetic interactions are more complicated: they still have electric charge, but through the Foldy–Wouthuysen transformation they acquire a $\boldsymbol{\mu}\cdot\boldsymbol{B}$ *magnetic moment* interaction, and (less well-known) an *electric charge radius* (manifested in the "Darwin term" in the Hamiltonian; *see* [9, 10, 11, 12]).

Acknowledgments

Helpful discussions with I. Khriplovich, J. Anandan, G. I. Opat, J. W. G. Wignall, A. J. Davies, M. J. Thomson, T. D. Kieu, D.-D. Wu and S. Bass are gratefully acknowledged. This work was supported in part by the Australian Research Council, an Australian Postgraduate Research Allowance and a Dixson Research Scholarship. We warmly thank the Institute for Nuclear Theory at the University of Washington for its hospitality and the United States Department of Energy Grant #DOE/ER40561 for partial support during the progress of this work.

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* This paper has been taken from Section 4.4.1 of JPC's Ph.D. thesis [1].

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