# Analytical Proof that $g_A \to 0$ in the Ultra-Relativistic Limit for the Harmonic Oscillator Relativistic Constituent Quark Model *

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## Abstract

We show analytically that $g_A \to 0$ in the ultrarelativistic limit for the harmonic oscillator relativistic constituent quark model.

## I. PROOF

Our notation essentially follows Berestetskii and Terent'ev [2–4]. Upon application of the Melosh transformation [5] one finds that $g_A$ is reduced from its non-relativistic value by a factor

$$Z \equiv \int d\Gamma \left|\Phi(M_0^2)\right|^2 \left\{1 - 2\,\frac{Q_\perp^2}{Q_\perp^2 + [m + (1-\eta)M_0]^2}\right\}.$$

(This same result—without the factor of 2 in the second term—holds for the reduction in electric dipole moment [1] and the contribution of the quark anomalous magnetic moment [6].) We parametrise the harmonic oscillator potential by

$$\Phi(M_0^2) = A \exp\left(-\frac{M_0^2}{12\alpha^2}\right).$$

The ultra-relativistic limit, $\alpha/m \to \infty$, can be realised here by taking $m = 0$ with $\alpha$ fixed. We then have

$$Z = 1 - 2\,\frac{I}{N} \tag{1}$$

where

$$I \equiv \frac{4}{(2\pi)^2} \int \frac{d^2q_\perp\, d\xi}{\xi(1-\xi)}\,\frac{d^2Q_\perp\, d\eta}{\eta(1-\eta)}\,\frac{Q_\perp^2}{Q_\perp^2 + (1-\eta)^2\left(\dfrac{Q_\perp^2}{\eta(1-\eta)} + \dfrac{q_\perp^2}{\eta\xi(1-\xi)}\right)}$$
$$\times\ \exp\left\{-\frac{Q_\perp^2}{12\alpha^2\eta(1-\eta)} - \frac{q_\perp^2}{12\alpha^2\eta\xi(1-\xi)}\right\}$$

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*This paper is taken directly from Appendix D of ref. [1].

and

$$N \equiv \frac{4}{(2\pi)^2} \int \frac{d^2q_\perp \, d\xi}{\xi(1-\xi)} \frac{d^2Q_\perp \, d\eta}{\eta(1-\eta)} \exp\left\{ -\frac{Q_\perp^2}{12\alpha^2\eta(1-\eta)} - \frac{q_\perp^2}{12\alpha^2\eta\xi(1-\xi)} \right\}.$$

The quantity $N$ takes care of the normalisation of $\Phi$; the common factors of $4/(2\pi)^2$ are introduced for later convenience. Converting $Q_\perp$ and $q_\perp$ to plane polar coördinates, the angular integrations are trivially equal to $2\pi$, and hence

$$I = 4\int \frac{qQ\,dq\,dQ\,d\xi\,d\eta}{\xi(1-\xi)\eta(1-\eta)} \frac{Q^2}{\dfrac{Q^2}{\eta} + \dfrac{(1-\eta)^2q^2}{\eta\xi(1-\xi)}} \exp\left( -\frac{Q^2}{12\alpha^2\eta(1-\eta)} - \frac{q^2}{12\alpha^2\eta\xi(1-\xi)} \right),$$

$$N = 4\int \frac{qQ\,dq\,dQ\,d\xi\,d\eta}{\xi(1-\xi)\eta(1-\eta)} \exp\left( -\frac{Q^2}{12\alpha^2\eta(1-\eta)} - \frac{q^2}{12\alpha^2\eta\xi(1-\xi)} \right),$$

where $Q \equiv |Q_\perp|$ and $q \equiv |q_\perp|$ are integrated from 0 to $\infty$. Now change to the variables

$$X \equiv Q^2, \qquad Y \equiv q^2,$$

and define the quantities

$$\beta \equiv 12\alpha^2\eta(1-\eta), \qquad \gamma \equiv 12\alpha^2\eta\xi(1-\xi),$$

to obtain

$$I = \int_0^1 d\xi \int_0^1 d\eta \int_0^\infty dX \int_0^\infty dY \, \exp\left( -\frac{X}{\beta} - \frac{Y}{\gamma} \right) \frac{1}{\xi(1-\xi)\eta(1-\eta)} \left( \frac{1}{\eta} + \frac{(1-\eta)^2}{\eta\xi(1-\xi)} \frac{Y}{X} \right)^{-1},$$

$$N = \left\{ \int_0^1 d\xi \int_0^1 d\eta \frac{1}{\xi(1-\xi)\eta(1-\eta)} \right\} \cdot \left\{ \int_0^\infty \exp\left( -\frac{X}{\beta} \right) dX \right\} \cdot \left\{ \int_0^\infty \exp\left( -\frac{Y}{\gamma} \right) dY \right\}.$$

The $X$ and $Y$ integrals for $N$ can be done immediately, leaving

$$N = \int_0^1 d\xi \int_0^1 d\eta \, \frac{\beta\gamma}{\xi(1-\xi)\eta(1-\eta)};$$

substituting back in the values of $\beta$ and $\gamma$ gives

$$N = 144\alpha^4 \int_0^1 d\xi \int_0^1 d\eta \, \eta.$$

$N$ can now be finished off completely, with the $\xi$-integration giving 1, and the $\eta$-integration giving 1/2, yielding

$$N = 72\alpha^4. \tag{2}$$

Returning, now, to $I$, concentrate on the $X$ and $Y$ integrations first. To this end, define the quantities

$$
\begin{aligned}
\delta &\equiv \frac{1}{\eta}, \\
\varepsilon &\equiv \frac{(1-\eta)^2}{\eta\xi(1-\xi)}, \\
\phi &\equiv \frac{1}{\xi(1-\xi)\eta(1-\eta)},
\end{aligned}
$$

which simplifies the integral to

$$
I = \int_0^1 d\xi \int_0^1 d\eta \int_0^\infty dX \int_0^\infty dY \, \frac{\phi}{\delta + \varepsilon \dfrac{Y}{X}} \, \exp\left( -\frac{X}{\beta} - \frac{Y}{\gamma} \right).
$$

Concentrating on the $Y$-integral,

$$
I_Y \equiv \int_0^\infty dY \, \frac{\exp\left( \dfrac{-Y}{\gamma} \right)}{\delta + \varepsilon \dfrac{Y}{X}},
$$

define a new variable

$$
W \equiv \frac{1}{\gamma} \left\{ Y + \frac{\delta X}{\varepsilon} \right\},
$$

which gives [7, Eq. 5.1.1]

$$
\begin{aligned}
I_Y &= \frac{X}{\varepsilon} \exp\left( \frac{\delta X}{\gamma \varepsilon} \right) \int_{\frac{\delta X}{\gamma\varepsilon}}^\infty \frac{e^{-W}}{W} dW \\
&\equiv \frac{X}{\varepsilon} \exp\left( \frac{\delta X}{\gamma \varepsilon} \right) E_1\left( \frac{\delta X}{\gamma \varepsilon} \right),
\end{aligned}
$$

where $E_1(z)$ is the exponential integral. Defining another two convenient quantities

$$
\begin{aligned}
\zeta &\equiv \frac{\delta}{\gamma\varepsilon}, \\
\omega &\equiv \frac{1}{\beta\zeta} - 1,
\end{aligned}
$$

one can now simplify the format of the $X$-integral:

$$
I_X \equiv \frac{1}{\varepsilon \zeta^2} \int_0^\infty dX \, e^{-\omega X} X \, E_1(X).
$$

It will be noted that $I_X$ is now in the form of a Laplace transform, *i.e.*

$$
\tilde{f}(s) \equiv \mathcal{L}\left\{ f(t) \right\} \equiv \int_0^\infty dt \, e^{-st} f(t).
$$

Using the fact the Laplace transform of the exponential integral is given by [8, Eq. 19.1]

$$\mathcal{L}\Big\{E_1(t)\Big\} = \frac{1}{s}\ln(s+1),$$

and that

$$\mathcal{L}\Big\{t\,f(t)\Big\} = -\frac{d}{ds}\mathcal{L}\Big\{f(t)\Big\},$$

one then finds that

$$I_X = \frac{\ln(\omega+1)}{\varepsilon\zeta^2\omega^2} - \frac{1}{\omega(\omega+1)}.$$

Expanding the quantities $\omega$, $\zeta$ and $\varepsilon$ back out in terms of $\xi$ and $\eta$, one has

$$\varepsilon \equiv \frac{(1-\eta)^2}{\eta\xi(1-\xi)},$$
$$\zeta \equiv \frac{1}{12\alpha^2\eta(1-\eta)^2},$$
$$\omega \equiv -\eta,$$

giving

$$I_X = 144\alpha^4\xi\eta^3(1-\xi)(1-\eta)^2\left\{\frac{\ln(1-\eta)}{\eta^2} + \frac{1}{\eta(1-\eta)}\right\}.$$

Returning now to the full expression including the $\xi$ and $\eta$ integrals, we have

$$I = 144\alpha^4\int_0^1 d\xi\int_0^1 d\eta\,\eta^2(1-\eta)\left\{\frac{\ln(1-\eta)}{\eta^2} + \frac{1}{\eta(1-\eta)}\right\}.$$

As $\xi$ has now dropped out of our equations, it can be trivially integrated to yield a factor of 1. Expanding out the braces, one finds that

$$I = 144\alpha^4\int_0^1 (1-\eta)\ln(1-\eta)\,d\eta + 144\alpha^4\int_0^1 \eta\,d\eta.$$

Noting that

$$\frac{d}{dx}\left\{-\frac{1}{2}(1-\eta)^2\ln(1-\eta) + \frac{1}{4}(1-\eta)^2\right\} = (1-\eta)\ln(1-\eta),$$

one finally obtains the desired result

$$I = 36\alpha^4. \qquad (3)$$

Inserting (2) and (3) into (1) gives the final result of

$$Z = 1 - 2\,\frac{36\alpha^4}{72\alpha^4}$$
$$= 0.$$

(It should be noted that, for the equivalent electric dipole moment and anomalous magnetic moment calculations, the result is 1/2, rather than zero.)

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