Analytical Proof that \( g_A \to 0 \) in the Ultra-Relativistic Limit for the Harmonic Oscillator Relativistic Constituent Quark Model *

John P. Costella and Bruce H. J. McKellar
School of Physics, The University of Melbourne, Parkville, Victoria 3052, Australia
(6 February 1994)

Abstract

We show analytically that \( g_A \to 0 \) in the ultrarelativistic limit for the harmonic oscillator relativistic constituent quark model.

I. PROOF

Our notation essentially follows Berestetskii and Terent’ev [2–4]. Upon application of the Melosh transformation [5] one finds that \( g_A \) is reduced from its non-relativistic value by a factor

\[
Z \equiv \int \frac{d\Gamma}{\Phi(M_0^2)} \left\{ 1 - 2 \frac{Q_{\perp}^2}{Q_{\perp}^2 + [m + (1 - \eta)M_0]^2} \right\}.
\]

(This same result—without the factor of 2 in the second term—holds for the reduction in electric dipole moment [1] and the contribution of the quark anomalous magnetic moment [6].) We parametrise the harmonic oscillator potential by

\[ \Phi(M_0^2) = A \exp \left( -\frac{M_0^2}{12\alpha^2} \right). \]

The ultra-relativistic limit, \( \alpha/m \to \infty \), can be realised here by taking \( m = 0 \) with \( \alpha \) fixed. We then have

\[
Z = 1 - 2 \frac{I}{N} \tag{1}
\]

where

\[
I \equiv \frac{4}{(2\pi)^2} \int \frac{d^2q_{\perp} \, d\xi}{\xi(1 - \xi) \eta(1 - \eta)} \int \frac{d^2Q_{\perp} \, d\eta}{Q_{\perp}^2 + (1 - \eta)^2 \left( \frac{Q_{\perp}^2}{\eta(1 - \eta)} + \frac{q_{\perp}^2}{\eta\xi(1 - \xi)} \right)} \times \exp \left\{ -\frac{Q_{\perp}^2}{12\alpha^2\eta(1 - \eta)} - \frac{q_{\perp}^2}{12\alpha^2\eta\xi(1 - \xi)} \right\}
\]

*This paper is taken directly from Appendix D of ref. [1].
and
\[ N \equiv \frac{4}{(2\pi)^2} \int d^2q \, d\xi \, d^2Q \, d\eta \, \exp \left\{ -\frac{Q^2}{12\alpha^2 \eta(1-\eta)} - \frac{q^2}{12\alpha^2 \eta \xi(1-\xi)} \right\} . \]

The quantity \( N \) takes care of the normalisation of \( \Phi \); the common factors of \( 4/(2\pi)^2 \) are introduced for later convenience. Converting \( Q \) and \( q \) to plane polar coordinates, the angular integrations are trivially equal to \( 2\pi \), and hence
\[ I = 4 \int qQ \, dq \, dQ \, d\xi \, d\eta \, \xi(1-\xi) \eta(1-\eta) \exp \left( -\frac{Q^2}{12\alpha^2 \eta(1-\eta)} - \frac{q^2}{12\alpha^2 \eta \xi(1-\xi)} \right) , \]
\[ N = 4 \int qQ \, dq \, dQ \, d\xi \, d\eta \, \xi(1-\xi) \eta(1-\eta) \exp \left( -\frac{Q^2}{12\alpha^2 \eta(1-\eta)} - \frac{q^2}{12\alpha^2 \eta \xi(1-\xi)} \right) , \]

where \( Q \equiv |Q_\perp| \) and \( q \equiv |q_\perp| \) are integrated from 0 to \( \infty \). Now change to the variables
\[ X \equiv Q^2 , \]
\[ Y \equiv q^2 , \]

and define the quantities
\[ \beta \equiv 12\alpha^2 \eta(1-\eta) , \]
\[ \gamma \equiv 12\alpha^2 \eta \xi(1-\xi) , \]

to obtain
\[ I = \int_0^1 d\xi \int_0^1 d\eta \int_0^\infty dX \int_0^\infty dY \exp \left( -\frac{X}{\beta} - \frac{Y}{\gamma} \right) \frac{1}{\xi(1-\xi) \eta(1-\eta)} \left( \frac{1}{\eta} + \frac{(1-\eta)^2}{\eta \xi(1-\xi)} \right) X^{-1} , \]
\[ N = \left\{ \int_0^1 d\xi \int_0^1 d\eta \frac{1}{\xi(1-\xi) \eta(1-\eta)} \right\} \cdot \left\{ \int_0^\infty \exp \left( -\frac{X}{\beta} \right) dX \right\} \cdot \left\{ \int_0^\infty \exp \left( -\frac{Y}{\gamma} \right) dY \right\} . \]

The \( X \) and \( Y \) integrals for \( N \) can be done immediately, leaving
\[ N = \int_0^1 d\xi \int_0^1 d\eta \frac{\beta \gamma}{\xi(1-\xi) \eta(1-\eta)} ; \]
substituting back in the values of \( \beta \) and \( \gamma \) gives
\[ N = 144\alpha^4 \int_0^1 d\xi \int_0^1 d\eta \eta . \]

\( N \) can now be finished off completely, with the \( \xi \)-integration giving 1, and the \( \eta \)-integration giving 1/2, yielding
\[ N = 72\alpha^4 . \] (2)

Returning, now, to \( I \), concentrate on the \( X \) and \( Y \) integrations first. To this end, define the quantities

\[ 2 \]
δ ≡ \frac{1}{\eta},

ε \equiv \frac{(1 - \eta)^2}{\eta \xi (1 - \xi)},

ϕ \equiv \frac{1}{\xi (1 - \xi) \eta (1 - \eta)},

which simplifies the integral to

\[ I = \int_0^1 d\xi \int_0^1 d\eta \int_0^\infty dX \int_0^\infty dY \frac{\phi}{\delta + \varepsilon} \exp \left( -\frac{X}{\beta} - \frac{Y}{\gamma} \right). \]

Concentrating on the Y-integral,

\[ I_Y \equiv \int_0^\infty dY \frac{\exp \left( -\frac{Y}{\gamma} \right)}{\delta + \varepsilon} \]

define a new variable

\[ W \equiv \frac{1}{\gamma} \left\{ Y + \frac{\delta X}{\varepsilon} \right\}, \]

which gives [7, Eq. 5.1.1]

\[ I_Y = \frac{X}{\varepsilon} \exp \left( \frac{\delta X}{\gamma \varepsilon} \right) \int_0^\infty e^{-W} dW \]

\[ = \frac{X}{\varepsilon} \exp \left( \frac{\delta X}{\gamma \varepsilon} \right) E_1 \left( \frac{\delta X}{\gamma \varepsilon} \right), \]

where \( E_1(z) \) is the exponential integral. Defining another two convenient quantities

\[ \zeta \equiv \frac{\delta}{\gamma \varepsilon}, \]
\[ \omega \equiv \frac{1}{\beta \zeta} - 1, \]

one can now simplify the format of the X-integral:

\[ I_X \equiv \frac{1}{\varepsilon \zeta^2} \int_0^\infty dX e^{-\omega X} E_1(X). \]

It will be noted that \( I_X \) is now in the form of a Laplace transform, i.e.

\[ \hat{f}(s) \equiv \mathcal{L}\{f(t)\} \equiv \int_0^\infty dt e^{-st} f(t). \]

Using the fact the Laplace transform of the exponential integral is given by [8, Eq. 19.1]
\[ \mathcal{L}\left\{ E_1(t) \right\} = \frac{1}{s} \ln(s + 1), \]

and that
\[ \mathcal{L}\left\{ t f(t) \right\} = -\frac{d}{ds}\mathcal{L}\left\{ f(t) \right\}, \]

one then finds that
\[ I_X = \frac{\ln(\omega + 1)}{\varepsilon \zeta^2 \omega^2} - \frac{1}{\omega(\omega + 1)}. \]

Expanding the quantities \( \omega, \zeta \) and \( \varepsilon \) back out in terms of \( \xi \) and \( \eta \), one has
\[ \varepsilon \equiv \frac{(1 - \eta)^2}{\eta \xi (1 - \xi)}, \]
\[ \zeta \equiv \frac{1}{12 \alpha^2 \eta (1 - \eta)^2}, \]
\[ \omega \equiv -\eta, \]

giving
\[ I_X = 144 \alpha^4 \xi \eta^3 (1 - \xi) (1 - \eta)^2 \left\{ \frac{\ln(1 - \eta)}{\eta^2} + \frac{1}{\eta(1 - \eta)} \right\}. \]

Returning now to the full expression including the \( \xi \) and \( \eta \) integrals, we have
\[ I = 144 \alpha^4 \int_0^1 d\xi \int_0^1 d\eta \eta^2 (1 - \eta) \left\{ \frac{\ln(1 - \eta)}{\eta^2} + \frac{1}{\eta(1 - \eta)} \right\}. \]

As \( \xi \) has now dropped out of our equations, it can be trivially integrated to yield a factor of 1. Expanding out the braces, one finds that
\[ I = 144 \alpha^4 \int_0^1 (1 - \eta) \ln(1 - \eta) \, d\eta + 144 \alpha^4 \int_0^1 \eta \, d\eta. \]

Noting that
\[ \frac{d}{dx} \left\{ -\frac{1}{2} (1 - \eta)^2 \ln(1 - \eta) + \frac{1}{4} (1 - \eta)^2 \right\} = (1 - \eta) \ln(1 - \eta), \]

one finally obtains the desired result
\[ I = 36 \alpha^4. \] (3)

Inserting (2) and (3) into (1) gives the final result of
\[ Z = 1 - 2 \frac{36 \alpha^4}{72 \alpha^4} = 0. \]

(It should be noted that, for the equivalent electric dipole moment and anomalous magnetic moment calculations, the result is \( 1/2 \), rather than zero.)
REFERENCES